1 **a** $0 \leq x \leq 12\pi \Leftrightarrow 0 \leq \frac{x}{3} \leq 4\pi$

$$\sin\left(\frac{x}{3}\right) = \frac{1}{2}$$
$$\frac{x}{3} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$
$$x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2}$$

b $-\pi \leq x \leq \pi \Leftrightarrow -2\pi \leq 2x \leq 2\pi$

$$\Leftrightarrow -2\pi + \frac{\pi}{6} \leq 2x + \frac{\pi}{6} \leq 2\pi + \frac{\pi}{6}$$
$$\Leftrightarrow -\frac{11\pi}{6} \leq 2x + \frac{\pi}{6} \leq \frac{13\pi}{6}$$
$$\sqrt{2}\cos\left(2x + \frac{\pi}{6}\right) + 1 = 0$$
$$\cos\left(2x + \frac{\pi}{6}\right) = -\frac{1}{\sqrt{2}}$$
$$2x + \frac{\pi}{6} = -\frac{5\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$$
$$2x + \frac{2\pi}{12} = -\frac{15\pi}{12}, -\frac{9\pi}{12}, \frac{9\pi}{12}, \frac{15\pi}{12}$$
$$2x = -\frac{17\pi}{12}, -\frac{11\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}$$
$$x = -\frac{17\pi}{24}, -\frac{11\pi}{24}, \frac{7\pi}{24}, \frac{13\pi}{24}$$

2

$$\cos(x) = \frac{\sqrt{3}}{2}$$
$$x = 2n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$$

3 **a** Since

$$\cos^2 A + \sin^2 A = 1,$$

we see that $\cos^2 A = 1 - \sin^2 A$
$$= 1 - \left(\frac{3}{5}\right)^2$$
$$= 1 - \frac{9}{25}$$
$$= \frac{16}{25}.$$

Therefore, $\cos A = \pm\frac{4}{5}$. However, as A is acute, we can reject the negative solution, giving $\cos A = \frac{4}{5}$. Therefore,

$$\sec A = \frac{1}{\cos A} = \frac{5}{4}.$$

b Using the result from the previous question we have, $\cot A = \frac{\cos A}{\sin A}$
$$= \frac{\frac{4}{5}}{\frac{3}{5}}$$
$$= \frac{4}{3}.$$

c Since

$$\cos^2 B + \sin^2 B = 1,$$

we see that $\sin^2 B = 1 - \cos^2 A$

$$= 1 - \left(-\frac{1}{2}\right)^2$$

$$= 1 - \frac{1}{4}$$

$$= \frac{3}{4}.$$

Therefore, $\sin B = \pm\frac{\sqrt{3}}{2}$. However, as B is obtuse, we can reject the negative, giving $\sin B = \frac{\sqrt{3}}{2}$. It follows that, $\cot B = \frac{\cos A}{\sin A}$

$$= \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= -\frac{\sqrt{3}}{3}.$$

d Using work from the previous question, we have, $\operatorname{cosec} B = \frac{1}{\sin B}$

$$= \frac{1}{\frac{\sqrt{3}}{2}}$$

$$= \frac{2\sqrt{3}}{3}.$$

4 Since $\cos A = 2\cos^2\frac{A}{2} - 1$, we know that $2\cos^2\frac{A}{2} - 1 = \cos A$

$$2\cos^2\frac{A}{2} - 1 = \frac{1}{3}$$

$$2\cos^2\frac{A}{2} = \frac{4}{3}$$

$$\cos^2\frac{A}{2} = \frac{2}{3}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{2}{3}}$$

Or equivalently, $\cos\frac{A}{2} = \pm\frac{\sqrt{6}}{3}$.

5 We have, $\text{LHS} = \frac{1}{1+\sin A} + \frac{1}{1-\sin A}$

$$= \frac{1-\sin A}{(1+\sin A)(1-\sin A)}$$

$$+ \frac{1+\sin A}{(1+\sin A)(1-\sin A)}$$

$$= \frac{2}{1-\sin^2 A}$$

$$= \frac{2}{\cos^2 A}$$

$$= 2\sec^2 A$$

$$= \text{RHS}.$$

6 **a** $\frac{1}{2}\sin(4x) - \frac{1}{2}\sin(2x)$

b $\theta=\frac{(2n+1)\pi}{2}$ or $\theta=\frac{(2n+1)\pi}{4}, n\in\mathbb{Z}$

7 **a**
$$\begin{aligned}\text{LHS}&=\frac{\sin 3x+\sin x}{\cos 3x+\cos x}\\&=\frac{2\sin 2x\cos x}{2\cos 2x\cos x}\\&=\frac{\sin 2x}{\cos 2x}\\&=\tan 2x\\&=\text{RHS}\end{aligned}$$

b
$$\begin{aligned}\text{LHS}&=\frac{\sin x+\sin 2x}{1+\cos x+\cos 2x}\\&=\frac{2\sin x\cos x+\sin x}{2\cos^x+\cos x}\\&=\frac{\sin x(2\cos x+1)}{\cos x(2\cos x+1)}\\&=\tan x\\&=\text{RHS}\end{aligned}$$

8 **a**
$$\begin{aligned}w+z&=(3+2i)+(3-2i)\\&=6\end{aligned}$$

b
$$\begin{aligned}w-z&=(3+2i)-(3-2i)\\&=3+2i-3+2i\\&=4i\end{aligned}$$

c
$$\begin{aligned}wz&=(3+2i)(3-2i)\\&=3^2-(2i)^2\\&=9+4\\&=13\end{aligned}$$

d
$$\begin{aligned}w^2+z^2&=(3+2i)^2+(3-2i)^2\\&=9+12i+(2i)^2+9-12i+(2i)^2\\&=18+4i^2+4i^2\\&=10\end{aligned}$$

e Using a previous result, we see that
$$\begin{aligned}(w+z)^2&=6^2\\&=36.\end{aligned}$$

f Using a previous result, we see that
$$\begin{aligned}(w-z)^2&=(4i)^2\\&=-16.\end{aligned}$$

g
$$\begin{aligned}w^2-z^2&=(w-z)(w+z)\\&=4i\times 6\\&=24i\end{aligned}$$

h Using the previous question,
$$\begin{aligned}(w-z)(w+z)&=w^2-z^2\\&=24i\end{aligned}$$

9 a

$$\begin{aligned} w+z &= (1-2i)+(2-3i) \\ &= 3-5i \end{aligned}$$

b

$$\begin{aligned} w-z &= (1-2i)-(2-3i) \\ &= 1-2i-2+3i \\ &= -1+i \end{aligned}$$

c

$$\begin{aligned} wz &= (1-2i)(2-3i) \\ &= 2-3i-4i+6i^2 \\ &= 2-7i-6 \\ &= -4-7i \end{aligned}$$

d

$$\begin{aligned} \frac{w}{z} &= \frac{1-2i}{2-3i} \\ &= \frac{(1-2i)}{(2-3i)}\frac{(2+3i)}{(2+3i)} \\ &= \frac{2+3i-4i-6i^2}{2^2-(3i)^2} \\ &= \frac{2-i-6}{4+9} \\ &= \frac{8-i}{13} \end{aligned}$$

e

$$\begin{aligned} iw &= i(1-2i) \\ &= i-2i^2 \\ &= 2+i \end{aligned}$$

f

$$\begin{aligned} \frac{i}{w} &= \frac{i}{1-2i} \\ &= \frac{i}{(1-2i)}\frac{(1+2i)}{(1+2i)} \\ &= \frac{i+2i^2}{1^2-(2i)^2} \\ &= \frac{-2+i}{1+4} \\ &= \frac{-2+i}{5} \end{aligned}$$

g

$$\begin{aligned} \frac{w}{i} &= \frac{1-2i}{i} \\ &= \frac{1-2i}{i}\frac{i}{i} \\ &= \frac{i-2i^2}{i^2} \\ &= \frac{2+i}{-1} \\ &= -2-i \end{aligned}$$

h

$$\begin{aligned}\frac{z}{w} &= \frac{2-3i}{1-2i} \\ &= \frac{(2-3i)}{(1-2i)}\frac{(1+2i)}{(1+2i)} \\ &= \frac{2+4i-3i-6i^2}{1^2-(2i)^2} \\ &= \frac{2+i+6}{1+4} \\ &= \frac{8+i}{5}\end{aligned}$$

i

$$\begin{aligned}\frac{w}{w+z} &= \frac{1-2i}{3-5i} \\ &= \frac{(1-2i)}{(3-5i)}\frac{(3+5i)}{(3+5i)} \\ &= \frac{3+5i-6i-10i^2}{3^2-(5i)^2} \\ &= \frac{3-i+10}{9+25} \\ &= \frac{13-i}{34}\end{aligned}$$

j

$$\begin{aligned}(1+i)w &= (1+i)(1-2i) \\ &= 1-2i+i-2i^2 \\ &= 1-i+2 \\ &= 3-i\end{aligned}$$

k

$$\begin{aligned}\frac{w}{1+i} &= \frac{1-2i}{1+i} \\ &= \frac{(1-2i)}{(1+i)}\frac{(1-i)}{(1-i)} \\ &= \frac{1-i-2i+2i^2}{1^2-i^2} \\ &= \frac{1-3i-2}{1+1} \\ &= \frac{-1-3i}{2}\end{aligned}$$

l

$$\begin{aligned}w^2 &= (1-2i)^2 \\ &= 1-4i+(2i)^2 \\ &= 1-4i-4 \\ &= -3-4i\end{aligned}$$

10a

$$\begin{aligned}z^2+49 &= z^2-(7i)^2 \\ &= (z-7i)(z+7i)\end{aligned}$$

b Here, we must complete this square, giving,

$$\begin{aligned}z^2-2z+10 &= (z^2-2z+1)-1+10 \\ &= (z-1)^2+9 \\ &= (z-1)^2-(3i)^2 \\ &= (z-1-3i)(z-1+3i)\end{aligned}$$

c Here, we must complete this square. Factor out the 9 first, so that

$$\begin{aligned}9z^2-6z+5&=9(z^2-\frac{2}{3}z+\frac{5}{9})\\&=9\left(\left(z^2-\frac{2}{3}z+\frac{1}{9}\right)-\frac{1}{9}+\frac{5}{9}\right)\\&=9\left(\left(z-\frac{1}{3}\right)^2+\frac{4}{9}\right)\\&=9\left(\left(z-\frac{1}{3}\right)^2-\left(\frac{2}{3}i\right)^2\right)\\&=9\left(z-\frac{1}{3}-\frac{2}{3}i\right)\left(z-\frac{1}{3}+\frac{2}{3}i\right)\end{aligned}$$

d Here, we must complete this square. Factor out the 4 first, so that

$$\begin{aligned}4z^2+12z+13&=4(z^2+3z+\frac{13}{4})\\&=4\left(\left(z^2+3z+\frac{9}{4}\right)-\frac{9}{4}+\frac{13}{4}\right)\\&=4\left(\left(z+\frac{3}{2}\right)^2+1\right)\\&=4\left(\left(z+\frac{3}{2}\right)^2-i^2\right)\\&=4\left(z+\frac{3}{2}-i\right)\left(z+\frac{3}{2}+i\right)\end{aligned}$$

11a We need to find real numbers a and b such that

$$\begin{aligned}(a+ib)^2&=3+4i\\a^2+2abi+(ib)^2&=3+4i\\(a^2-b^2)+2abi&=3+4i\end{aligned}$$

Therefore,

$$\begin{aligned}a^2+b^2&=3\quad(1)\\2ab&=4\quad(2).\end{aligned}$$

Solving equation (2) for b gives $b=\frac{2}{a}$, and substituting into equation (1) gives

$$\begin{aligned}a^2+\left(\frac{2}{a}\right)^2&=3\\a^2-\frac{4}{a^2}&=3\\a^4-4&=3a^2\\a^4-3a^2-4&=0\\(a^2-4)(a^2+1)&=0\\a&=\pm2.\end{aligned}$$

And correspondingly, $b=\pm1$. Therefore the two square roots are $2+i$ and $-2-i$.

b We have

$$\begin{aligned}z&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\&=\frac{-(4+3i)\pm\sqrt{(4+3i)^2-4(2-i)(-1+3i)}}{2(2-i)}\\&=\frac{-4-3i\pm\sqrt{16+24i+(3i)^2-4(-2+6i+i-3i^2)}}{4-2i}\\&=\frac{-4-3i\pm\sqrt{16+24i-9-4(-2+7i+3)}}{4-2i}\\&=\frac{-4-3i\pm\sqrt{7+24i-4(1+7i)}}{4-2i}\end{aligned}$$

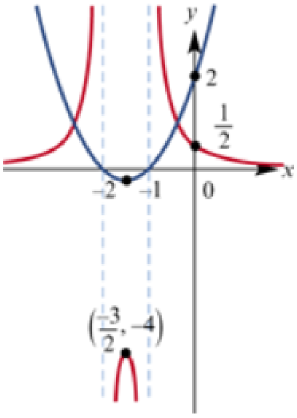
$$= \frac{-4-3i \pm \sqrt{7+24i-4-28i}}{4-2i}$$

$$= \frac{-4-3i \pm \sqrt{3-4i}}{4-2i}$$

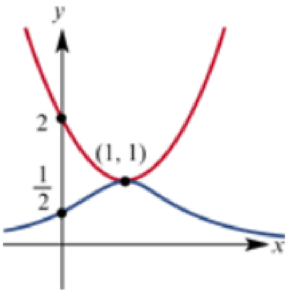
Show that $\sqrt{3-4i} = 2-i$

Therefore, $z = -i$ or $z = -1-i$

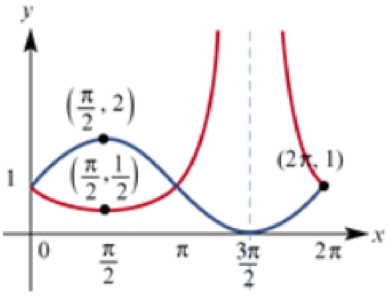
12a



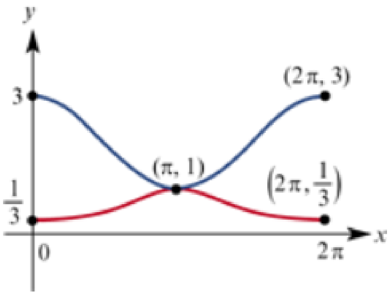
b



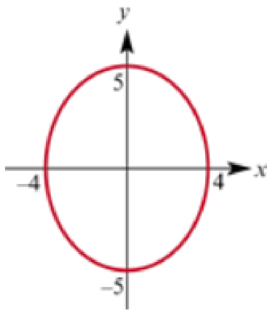
c



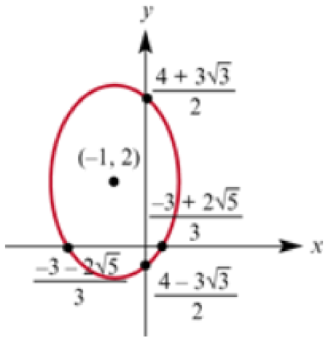
d



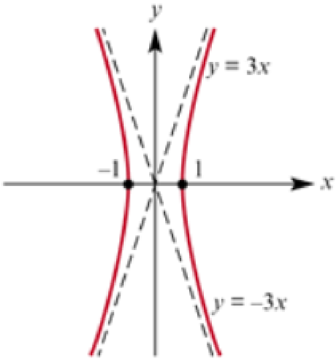
13a



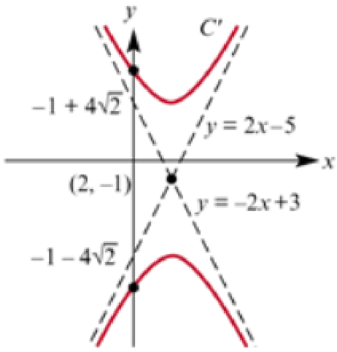
b



14a



b



15 We know that the point $P(x, y)$ satisfies,

$$\begin{aligned} AP &= BP \\ \sqrt{(x-2)^2+(y-2)^2} &= \sqrt{(x-3)^2+(y-4)^2} \\ (x-2)^2+(y-2)^2 &= (x-3)^2+(y-4)^2 \\ x^2-4x+4+y^2-4y+4 &= x^2-6x+9+y^2-8y+16 \\ -4x+4-4y+4 &= -6x+9-8y+16 \\ 2x+4y &= 17 \end{aligned}$$

16 Let (x, y) be the coordinates of point P. If $FP = \frac{1}{2}MP$ then

$$\sqrt{x^2+(y-1)^2} = \frac{1}{2}\sqrt{(x-(-3))^2}.$$

Squaring both sides gives

$$\begin{aligned} x^2+(y-1)^2 &= \frac{1}{4}(x+3)^2 \\ 4x^2+4(y-1)^2 &= x^2+6x+9 \\ 3x^2-6x+4(y-1)^2 &= 9 \\ 3x^2-6x+4(y-1)^2 &= 9 \end{aligned}$$

Completing the square

$$\begin{aligned} 3x^2-6x+4(y-1)^2 &= 9 \\ 3(x^2-2x)+4(y-1)^2 &= 9 \\ 3((x^2-2x+1)-1)+4(y-1)^2 &= 9 \\ 3((x-1)^2-1)+4(y-1)^2 &= 9 \\ 3(x-1)^2+4(y-1)^2 &= 12 \\ \text{or equivalently } \frac{(x-1)^2}{4}+\frac{(y-1)^2}{3} &= 1. \end{aligned}$$

This is an ellipse with centre $(1,1)$.

17 We know that the point $P(x,y)$ satisfies,

$$\begin{aligned} FP &= RP \\ \sqrt{x^2+(y-1)^2} &= \sqrt{(y-(-3))^2} \\ x^2+(y-1)^2 &= (y+3)^2 \\ x^2+y^2-2y+1 &= y^2+6y+9 \\ x^2-2y+1 &= 6y+9 \\ 8y &= x^2-8 \\ y &= \frac{x^2}{8}-1 \end{aligned}$$

Therefore, the set of points is a parabola whose equation is $y=\dfrac{x^2}{8}-1$

18a Since $x=2t+1$ and $y=2-3t$ we solve both equations for t to find that

$$t=\frac{x-1}{2} \text{ and } t=\frac{2-y}{3}.$$

Eliminating t then gives

$$\begin{aligned} \frac{x-1}{2} &= \frac{2-y}{3} \\ 3(x-1) &= 2(2-y) \\ 3x-3 &= 4-2y \\ 3x+2y &= 7. \end{aligned}$$

b Since $\begin{aligned} x^2+y^2 &= \cos^2 2t+\sin^2 2t \\ &= 1, \end{aligned}$ these equations parameterise a circle with centre $(0,0)$ and radius 1.

c Solving each equation for the $\cos t$ and $\sin t$ respectively gives,

$$\cos t=\frac{x-2}{2} \text{ and } \sin t=\frac{y-3}{3}.$$

Therefore,

$$\left(\frac{x-2}{2}\right)^2 + \left(\frac{y-3}{3}\right)^2 = \cos^2 t + \sin^2 t$$
$$\frac{(x-2)^2}{4} + \frac{(y-3)^2}{9} = 1.$$

d Solving each equation for the $\tan t$ and $\sec t$ respectively gives,

$$\tan t = \frac{x}{2} \text{ and } \sec t = \frac{y}{3}.$$

Therefore,

$$\left(\frac{y}{3}\right)^2 - \left(\frac{x}{2}\right)^2 = \sec^2 t - \tan^2 t$$
$$\frac{y^2}{9} - \frac{x^2}{4} = 1.$$

19a Since $x = t - 1$ we know that $t = x + 1$. Substitute this into the second parametric equation to give,

$$y = 1 - 2t^2$$
$$= 1 - 2(x+1)^2.$$

b Since $0 \leq t \leq 2$ we have

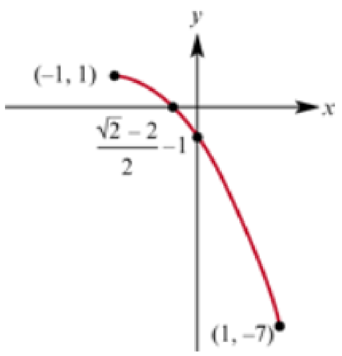
$$0 \leq x + 1 \leq 2$$
$$-1 \leq x \leq 1$$

c Since $0 \leq t \leq 2$, we have that

$$-7 \leq 1 - 2t^2 \leq 1.$$

Therefore the range is $-7 \leq y \leq 1$.

d



20 We have

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$= 2\cos 7\pi/6 \qquad = 2\sin 7\pi/6$$
$$= -\sqrt{3} \qquad = -1$$

so that the cartesian coordinates are $(-\sqrt{3}, -1)$.

21 Finding r first, gives,

$r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$. Since

$\tan\theta = \frac{-2}{2} = -1$, we can assume that $\theta = -\frac{\pi}{4}$ so that the point has polar coordinates $\left(2\sqrt{2}, -\frac{\pi}{4}\right)$. We could also let $r = -2\sqrt{2}$ and add π to the found angle, giving, $\left(-2\sqrt{2}, \frac{3\pi}{4}\right)$.

22a Since $r = 5$ and $r^2 = x^2 + y^2$ we know that $x^2 + y^2 = 5^2$. This is a circle of radius 5 centred at the origin.

b Since $\tan\theta = \frac{y}{x}$ we know that

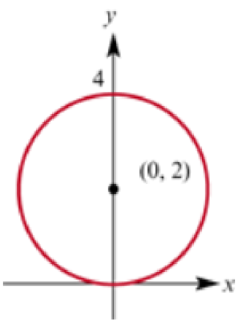
$$\frac{y}{x} = \tan\left(\frac{\pi}{3}\right)$$
$$\frac{y}{x} = \sqrt{3}$$
$$y = \sqrt{3}x.$$

c Since $y = r\sin\theta$, we know that
$$r = \frac{3}{\sin\theta}$$
$$r\sin\theta = 3$$
$$y = 3.$$

d Since $x = r\cos\theta$ and $y = r\sin\theta$, we know that
$$\frac{2}{3\sin\theta + 4\cos\theta} = r$$
$$3r\sin\theta + 4r\cos\theta = 2$$
$$3y + 4x = 2.$$

e Since $\sin(2\theta) = 2\sin\theta\cos\theta$, we have
$$r^2 = \frac{1}{\sin(2\theta)}$$
$$r^2\sin(2\theta) = 1$$
$$2r^2\sin\theta\cos\theta = 1$$
$$2(r\sin\theta)(r\cos\theta) = 1$$
$$2yx = 1$$
$$y = \frac{1}{2x}.$$

23a



b You can start with the polar equation and show that it has the given cartesian equation or visa versa. We start with $r = 4\sin\theta$. Multiplying both sides by r gives,
$$r^2 = 4r\sin\theta$$
$$x^2 + y^2 = 4x$$
$$x^2 - 4x + y^2 = 0$$
$$(x^2 - 4x + 4) - 4 + y^2 = 0$$
$$(x-2)^2 + y^2 = 2^2,$$
as required.